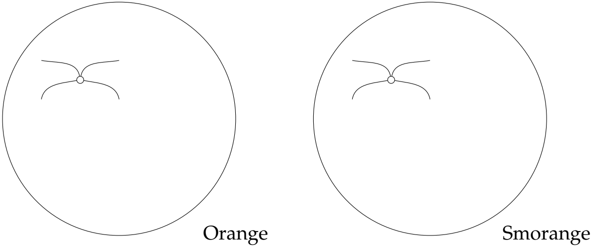
Turns out a smorange is just an orange

Sofic shifts

Charles T. Gray



OVERVIEW

Words, languages, and shifts (a reminder)

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Alphabet symbols

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Alternating strings: only words of length 2 are 01 and 10.

SOFIC SHIFTS

$$\begin{tikzcd} {} \arrow[r] & X \arrow[r, "\text{SFT}"] \arrow[d, "h"'] & X \arrow[r] \arrow[d, "h"] & {} \\ {} \arrow[r] & Y \arrow[r, "\text{sofic}"] & Y \arrow[r] & {} \end{tikzcd}$$

Figure: A sofic shift is the semi-conjugate image of a shift of finite type.

h must be continuous and onto, and the diagram above must commute.

REPRESENTING, THAT IS, PRESENTING

Shifts can be represented with edge-labelled digraphs. If an edge-labelled digraph G represents a shift X, we say G presents X.

Figure: An edge-labelled digraph presenting all strings from the alphabet $\{0, 1\}$. Thanks JB, the default loops *were* too small.

The set of infinite strings presented by a an edge-labelled digraph is denoted X_G.

Presentations of sofic shifts

Theorem 6.3 A shift X is sofic iff there exists an edge-labelled digraph G such that $X = X_G$.

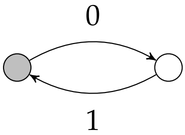

Figure: Petal graph presents the alternating set.

So, we have a finite representation of a potentially infinite collection of infinite strings. The word 'sofic' derives from the Hebrew word for finite.

Presentations are not unique

Graphs of alternating strings $\{010101\ldots, 101010\ldots\}$.

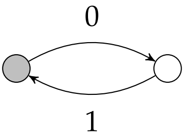

Figure: Petal and flower graphs present the alternating strings.

By adding ‘petal’s to the graph we can construct infinitely many presentations of the same set of strings.

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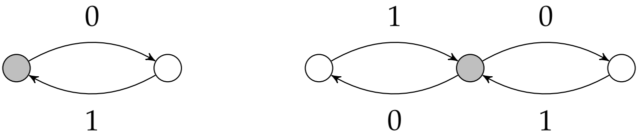

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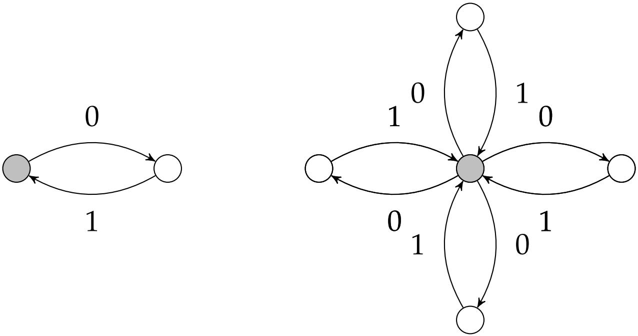

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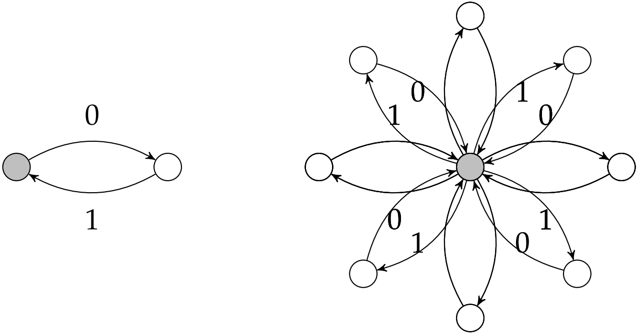

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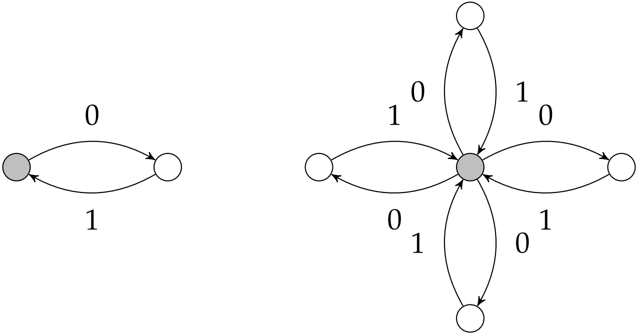

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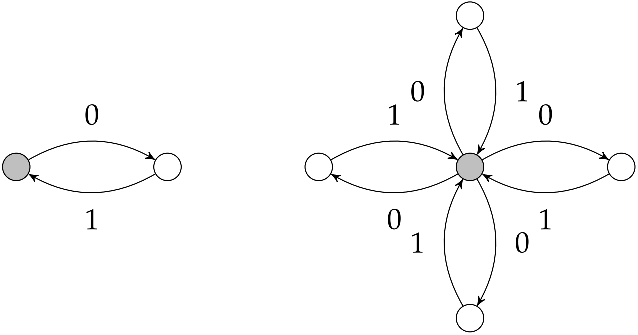

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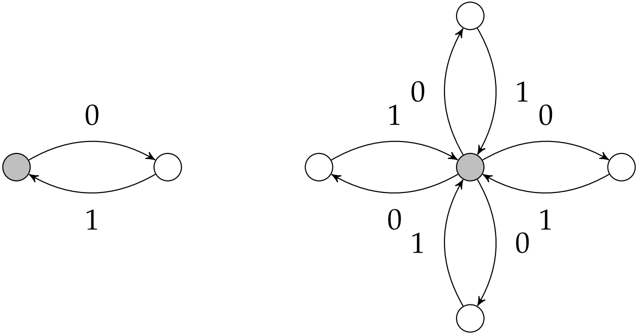


Figure: Petal and flower graphs present alternating strings.

- Least number of vertices.
- Only one edge with a given label out of each vertex.

A UNIQUE 'BEST' PRESENTATION OF A SOFIC SHIFT X

It turns out there is a unique 'best' presentation of X.

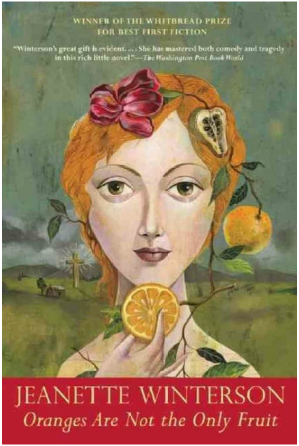
Theorem 6.8 (ahem, paraphrasing) If two graphs G and H present a sofic shift X and have the two characteristics

- least number of vertices and
- only one edge with a given label out of each vertex,

then the two graphs are isomorphic.

Such orange, very wow

SUCH ORANGE, VERY WOW



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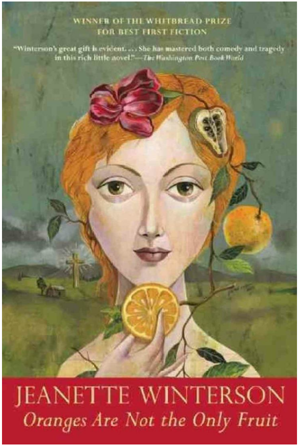

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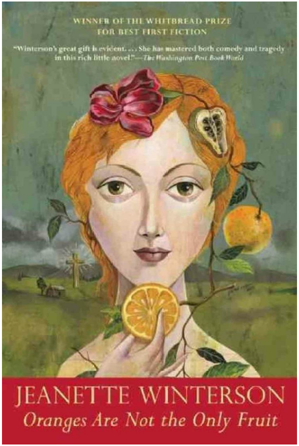

"Whoever said orange is the new pink was seriously disturbed." (Elle Woods, *Legally Blonde* 2001).

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Some very useful results (for 'best' presentations):

- If w is synchronising and focusses to I then $F(w) = F_G(I)$.

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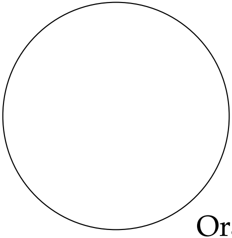
- If w is synchronising then for all $v \in F(w)$ we have wv synchronising.

Switch to one-note now and try to, as Sir Micheal would say, "prove some stuff".

MAKING AN ORANGE AND A SMORANGE

Orange and Smorange present a sofic shift X s.t.

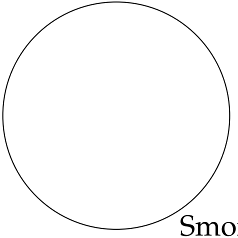
- least number of vertices
- only one edge with a given label out of each vertex.



Orange

Tools we need

- w a word $\implies \exists v$ s.t. wv synch.
- w synch. then wv synch $\forall v \in F(w)$



Smorange

THE SMORANGE IS JUST AN ORANGE IN DISGUISE

Orange and Smorange present a sofic shift X s.t.

- least number of vertices
- only one edge with a given label out of each vertex.

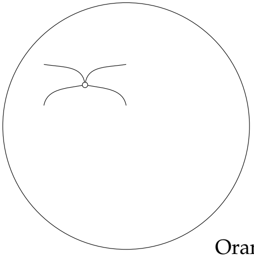
Tools we need

- $\exists$ synch. w in both graphs
- $\exists$ a path between any pair of vertices
- w synch. then wv synch $\forall v \in F(w)$
- $F_G(I) = F_G(J) \implies I = J$
- if w focusses to I, $F(w) = F_G(I)$

Important properties of ψ. For all vertices $I \in V(Orange)$

($\flat$) $F_{Orange}(I) = F_{Smorange}(\psi(I))$

($\sharp$) $\exists z \in L(X)$ s.t. z focusses to I



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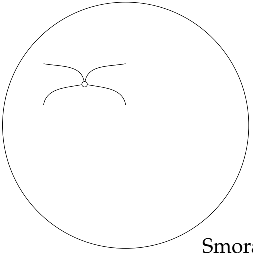

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